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LETTER TO THE EDITOR

Equivalence of TBA and QTM

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Abstract

The traditional thermodynamic Bethe ansatz equations for the *XXZ* model at $|\Delta| \geqslant 1$ are derived within the quantum transfer matrix method. This provides further evidence of the equivalence of both methods. Most importantly, we derive an integral equation for the free energy formulated for just one unknown function. This integral equation differs in physical and mathematical aspects to the established ones. The single integral equation is analytically continued to the regime $|\Delta| < 1$.

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1. Introduction

The thermodynamics of one-dimensional solvable models is generally determined by the solution to a set of so-called thermodynamic Bethe ansatz (TBA) equations [1]. Some lattice spin models such as the *XXZ* chain and *XYZ* chain have also been treated by the quantum transfer matrix (QTM) method [2–4, 6], see also chapters 17 and 18 of [1]. Correlated electron systems such as the *t–J* model and Hubbard model, have also been treated by the TBA and QTM methods [7–12].

The equations obtained by the QTM approach are quite different from those of the TBA. However, the numerical results of the two methods for the free energies are the same. Mathematically the nonlinear integral equations presented in [1] and [6] share similarities insofar as they can be interpreted as equations for dressed energies of elementary particles of magnon and spinon type, respectively.

Recently, from the TBA stand point one of the authors (MT) [13] derived in the case of the *XXZ* chain a simple integral equation for just one unknown function. This integral equation is completely different in structure from those mentioned above. Here we aim at a derivation of this equation in the QTM approach providing a more explicit as well as unified understanding of the structures and involved functions.

To be definite, we first consider the region $\Delta \geqslant 1$,

$$\mathcal{H} = -J \sum_{i=1}^{N} \left\{ S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta (S_i^z S_{i+1}^z - \tfrac{1}{4}) \right\} - 2h \sum_{i=1}^{N} S_i^z. \tag{1}$$

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The TBA equations for this model at temperature T are called the Gaudin–Takahashi equations, [14, 15]:

$$\begin{aligned}
\ln \eta_1(x) &= \frac{2\pi J \sinh\phi}{T\phi} s(x) + s * \ln(1+\eta_2(x)) \\
\ln \eta_j(x) &= s * \ln(1+\eta_{j-1}(x))(1+\eta_{j+1}(x)) \qquad j = 2, 3, \ldots \\
\lim_{l\to\infty} \frac{\ln \eta_l}{l} &= \frac{2h}{T}.
\end{aligned} \tag{2}$$

Here we put

$$\begin{aligned}
&\Delta = \cosh\phi \qquad Q \equiv \pi/\phi, \qquad s(x) = \frac{1}{4}\sum_{n=-\infty}^{\infty} \operatorname{sech}\left(\frac{\pi(x-2nQ)}{2}\right) \\
&s * f(x) \equiv \int_{-Q}^{Q} s(x-y) f(y)\,\mathrm{d}y.
\end{aligned} \tag{3}$$

The free energy per site is

$$\begin{aligned}
f &= \frac{2\pi J \sinh\phi}{\phi} \int_{-Q}^{Q} a_1(x) s(x)\,\mathrm{d}x - T \int_{-Q}^{Q} s(x) \ln(1+\eta_1(x))\,\mathrm{d}x \\
a_1(x) &\equiv \frac{\phi \sinh\phi/(2\pi)}{\cosh\phi - \cos(\phi x)}.
\end{aligned} \tag{4}$$

From this equation Takahashi [13] derived

$$\begin{aligned}
u(x) = 2\cosh\left(\frac{h}{T}\right) + \oint_C \frac{\phi}{2}\Bigg(&\cot\frac{\phi}{2}[x-y-2\mathrm{i}] \exp\left[-\frac{2\pi J \sinh\phi}{T\phi} a_1(y+\mathrm{i})\right] \\
&+ \cot\frac{\phi}{2}[x-y+2\mathrm{i}] \exp\left[-\frac{2\pi J \sinh\phi}{T\phi} a_1(y-\mathrm{i})\right]\Bigg) \frac{1}{u(y)} \frac{\mathrm{d}y}{2\pi\mathrm{i}}
\end{aligned} \tag{5}$$

where the free energy is given by

$$f = -T \ln u(0). \tag{6}$$

The contour C is an arbitrary counterclockwise closed loop around 0 where $2nQ$, $n \neq 0$ and $\pm 2\mathrm{i} + 2nQ$ should lie outside of this loop. Furthermore, this loop should not contain zeros of $u(y)$. It is expected that $u(y)$ has no zero in the region $|\mathrm{Im}\, y| \leqslant 1$. We show that these equations can be derived in the QTM approach.

2. Quantum transfer matrix and fusion hierarchy models

The QTM for this model is equivalent to that of the diagonal-to-diagonal transfer matrix of the six-vertex model which is a staggered or inhomogeneous row-to-row transfer matrix, see below. The partition function $Z \equiv \mathrm{Tr} \exp(-\mathcal{H}/T)$ is given by

$$\begin{aligned}
&Z = \sum_{\{\sigma\}} \prod_{j=1}^{N} \prod_{i=1}^{M} A(\sigma_{2i+j,j}\sigma_{2i+j+1,j};\, \sigma_{2i+j,j+1}\sigma_{2i+j+1,j+1}) \\
&A(\sigma_1\sigma_2; \sigma_1'\sigma_2') = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & c & b' & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \\
&a = \exp\left(-\frac{J\Delta}{2MT}\right) \sinh\left(\frac{J}{2MT}\right) \qquad b = \exp\left(\frac{-h}{MT}\right) \\
&b' = \exp\left(\frac{h}{MT}\right) \qquad c = \exp\left(-\frac{J\Delta}{2MT}\right) \cosh\left(\frac{J}{2MT}\right).
\end{aligned} \tag{7}$$

Then in the case $N = 2M \times$ integer, we have

$$
\begin{aligned}
&Z = \mathrm{Tr}\, \boldsymbol{T}^N \\
&\boldsymbol{T}(\sigma_1, \sigma_2, \dots, \sigma_{2M}; \sigma_1', \sigma_2', \dots, \sigma_{2M}') \\
&\qquad \equiv A(\sigma_1\sigma_2; \sigma_{2M}'\sigma_1') A(\sigma_3\sigma_4; \sigma_2'\sigma_3') \dots A(\sigma_{2M-1}\sigma_{2M}; \sigma_{2M-2}'\sigma_{2M-1}').
\end{aligned}
$$

The eigenvalue problem of this transfer matrix is a special case of the inhomogeneous six-vertex model on the square lattice.

Consider an inhomogeneous six-vertex model with the following column-dependent Boltzmann weights:

$$
\begin{aligned}
a_l &= \rho_l \mathsf{h}(v + v_l + \eta) \\
b_l &= \rho_l \omega^{-1} \mathsf{h}(v + v_l - \eta) \\
b_l' &= \rho_l \omega \mathsf{h}(v + v_l - \eta) \\
c_l &= \rho_l \mathsf{h}(2\eta) \qquad l = 1, \dots, L.
\end{aligned}
\tag{8}
$$

Here L is the number of columns, $\mathsf{h}(u)$ is u, $\sin(u)$ or $\sinh(u)$ depending on the anisotropy parameter. The transfer matrix $\boldsymbol{T}(v)$ acts in a 2^L-dimensional space,

$$
\begin{aligned}
&\boldsymbol{T} = \mathrm{Tr}\, \boldsymbol{R}_1(\sigma_1, \sigma_1') \boldsymbol{R}_2(\sigma_2, \sigma_2') \dots \boldsymbol{R}_L(\sigma_L, \sigma_L') \\
&\boldsymbol{R}_l(++) = \begin{pmatrix} a_l & 0 \\ 0 & b_l \end{pmatrix} \qquad \boldsymbol{R}_l(+-) = \begin{pmatrix} 0 & 0 \\ c_l & 0 \end{pmatrix} \\
&\boldsymbol{R}_l(-+) = \begin{pmatrix} 0 & c_l \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{R}_l(--) = \begin{pmatrix} b_l' & 0 \\ 0 & a_l \end{pmatrix}.
\end{aligned}
\tag{9}
$$

The space is divided into subspaces characterized by the number of down spins k. Without loss of generality we can put $k \leqslant L/2$. In this subspace we can construct Bethe-ansatz wavefunctions with k parameters $u_1, \dots, u_k$,

$$
\begin{aligned}
&|\Psi\rangle = \sum f(y_1, y_2, \dots, y_k) \sigma_{y_1}^- \sigma_{y_2}^- \dots \sigma_{y_k}^- |0\rangle \\
&f(y_1, y_2, \dots, y_k) = \sum_P A(P) \prod_{j=1}^{k} F(y_j; u_{Pj}) \\
&F(y; u) \equiv \omega^y \prod_{l=1}^{y-1} \mathsf{h}(u + v_l + \eta) \prod_{l=y+1}^{L} \mathsf{h}(u + v_l - \eta) \\
&A(P) = \epsilon(P) \sum_{j<l} \mathsf{h}(u_{Pj} - u_{Pl} - 2\eta).
\end{aligned}
\tag{10}
$$

Imposing periodic boundary conditions the Bethe ansatz equations (BAE) take the form

$$
\begin{aligned}
&\frac{\varphi(u_j + \eta)}{\varphi(u_j - \eta)} = -\omega^{-L} \prod_{m=1}^{k} \frac{\mathsf{h}(u_j - u_m + 2\eta)}{\mathsf{h}(u_j - u_m - 2\eta)} \\
&\varphi(v) = \prod_{l=1}^{L} \rho_l \mathsf{h}(v + v_l).
\end{aligned}
\tag{11}
$$

The corresponding eigenvalue of the transfer matrix is given by

$$
\begin{aligned}
&T_1(v) = \omega^{-L+k} \varphi(v - \eta) \frac{Q(v + 2\eta)}{Q(v)} + \omega^k \varphi(v + \eta) \frac{Q(v - 2\eta)}{Q(v)} \\
&Q(v) = \prod_{j=1}^{k} \mathsf{h}(v - u_j).
\end{aligned}
\tag{12}
$$

In order to solve the diagonal-to-diagonal transfer matrix we have to consider an inhomogeneous six-vertex model the Boltzmann weights of which are given by

$$\begin{aligned}
&a_l = c_l = 1 \qquad b_l = b_l' = 0 && \text{for even } l \\
&a_l = \exp\left(-\frac{J\Delta}{2MT}\right)\sinh\left(\frac{J}{2MT}\right) \qquad b_l = \exp\left(\frac{-h}{MT}\right) && \\
&b_l' = \exp\left(\frac{h}{MT}\right) \qquad c_l = \exp\left(-\frac{J\Delta}{2MT}\right)\cosh\left(\frac{J}{2MT}\right) && \text{for odd } l.
\end{aligned} \tag{13}$$

The conditions (8) are satisfied if we put

$$L = 2M \qquad \omega = \exp\left(\frac{h}{MT}\right) \qquad v = 0 \qquad \frac{\mathsf{h}'(2\eta)}{\mathsf{h}'(0)} = \frac{\sinh(\frac{J\Delta}{MT})}{\sinh(\frac{J}{MT})} \tag{14}$$

and

$$\begin{aligned}
&\rho_l = 1/\mathsf{h}(2\eta) \qquad v_l = \eta \qquad \text{for even } l \\
&\rho_l = \frac{\sqrt{bb'}}{\mathsf{h}(v_l - \eta)} \qquad \frac{\mathsf{h}(v_l + \eta)}{\mathsf{h}(v_l - \eta)} = \frac{a}{\sqrt{bb'}} = \exp\left(-\frac{J\Delta}{2MT}\right)\sinh\left(\frac{J}{2MT}\right) \\
&\qquad\qquad \text{for odd } l.
\end{aligned} \tag{15}$$

Putting $\eta + v_1 = 2\alpha_M$ we have

$$\varphi(v) = \left(\frac{\mathsf{h}(v+\eta)\mathsf{h}(v+2\alpha_M-\eta)}{\mathsf{h}(2\eta)\mathsf{h}(2\alpha_M-2\eta)}\right)^M. \tag{16}$$

The largest eigenvalue belongs to the $k = M$ sector. The BAE for $u_j,\ j = 1, \ldots, M$, are

$$\frac{\varphi(u_j+\eta)}{\varphi(u_j-\eta)} = -\mathrm{e}^{-2h/T}\prod_{m=1}^{M}\frac{\mathsf{h}(u_j-u_m+2\eta)}{\mathsf{h}(u_j-u_m-2\eta)}. \tag{17}$$

The corresponding eigenvalue is given by

$$T_1(v) = \mathrm{e}^{-h/T}\varphi(v-\eta)\frac{Q(v+2\eta)}{Q(v)} + \mathrm{e}^{h/T}\varphi(v+\eta)\frac{Q(v-2\eta)}{Q(v)}. \tag{18}$$

Due to the BAE (17), the eigenvalue $T_1(x)$ is an entire function in the complex plane. The free energy per site is given by

$$f = -T\lim_{M\to\infty}\ln T_1(0). \tag{19}$$

The matrix $T_1(v)$ can be embedded into a more general family of matrices provided by the fusion hierarchy [17],

$$T_j(v) \equiv \sum_{l=0}^{j}\mathrm{e}^{-(j-2l)h/T}\varphi(v-(j-2l)\eta)\frac{Q(v+(j+1)\eta)Q(v-(j+1)\eta)}{Q(v+(2l-j+1)\eta)Q(v+(2l-j-1)\eta)}. \tag{20}$$

The eigenvalues $T_j(v)$ as functions of v are all entire in the complex plane. It is easily seen that the following functional relations hold [17]:

$$\begin{aligned}
&T_j(v+\eta)T_j(v-\eta) = \varphi(v+(j+1)\eta)\varphi(v-(j+1)\eta) + T_{j+1}(v)T_{j-1}(v) \\
&T_0(v) \equiv \varphi(v).
\end{aligned} \tag{21}$$

3. Derivation of the Gaudin–Takahashi equation

For $\Delta > 1$ we put

$$\begin{aligned} &\mathsf{h}(u) = \sin u \qquad \eta = \mathrm{i}\tilde{\phi}/2 \qquad \tilde{\phi} = \cosh^{-1}\left(\frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)}\right) \\ &\alpha_M = \frac{\mathrm{i}}{2}\tanh^{-1}\left(\tanh\tilde{\phi}\tanh\frac{J\Delta}{2MT}\right). \end{aligned} \tag{22}$$

In the limit of $M \to \infty$ we have

$$\tilde{\phi} = \phi \qquad M\alpha_M = \mathrm{i}J\sinh\phi/(4T). \tag{23}$$

We transform the parameter v to $x \equiv \mathrm{i}v/\eta$. Then equations (16) and (20) turn into

$$\begin{aligned} &Q(x) = \prod_{j=1}^{M}\sin\frac{\tilde{\phi}}{2}(x - x_j) \qquad \varphi(x) = \left(\frac{\sin\frac{\tilde{\phi}}{2}(x+\mathrm{i})\sin\frac{\tilde{\phi}}{2}(x-(1-2u_M)\mathrm{i})}{\sinh\tilde{\phi}\sinh\tilde{\phi}(1-u_M)}\right)^M \\ &u_M = \alpha_M/\eta \qquad x_j = \mathrm{i}u_j/\eta \end{aligned} \tag{24}$$

$$T_j(x) \equiv \sum_{l=0}^{j}\mathrm{e}^{-(j-2l)h/T}\varphi(x-(j-2l)\mathrm{i})\frac{Q(x+(j+1)\mathrm{i})Q(x-(j+1)\mathrm{i})}{Q(x+(2l-j+1)\mathrm{i})Q(x+(2l-j-1)\mathrm{i})}. \tag{25}$$

These functions are all entire in the complex plane. Now we introduce a modified eigenvalue of $T_j(x)$

$$\tilde{T}_j(x) \equiv T_j(x)\left(\frac{\sinh(\tilde{\phi})\sinh\tilde{\phi}(1-u_M)}{\sin\frac{\tilde{\phi}}{2}(x+(j+1)\mathrm{i})\sin\frac{\tilde{\phi}}{2}(x-(j+1-2u_M)\mathrm{i})}\right)^M. \tag{26}$$

In contrast to the entire function $T_j(x)$, $\tilde{T}_j(x)$ has poles of order M at $x = 2nQ + u_M\mathrm{i} \pm (1 + j - u_M)\mathrm{i}$. On the other hand, it has constant asymptotics

$$\tilde{T}_j(\pm\mathrm{i}\infty) = \frac{\sinh(j+1)h/T}{\sinh h/T}. \tag{27}$$

From (21), we can find the following functional relation for $\tilde{T}_j(x)$:

$$\tilde{T}_j(x+\mathrm{i})\tilde{T}_j(x-\mathrm{i}) = \boldsymbol{b}_j(x) + \tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x) \tag{28}$$

where we have defined

$$\boldsymbol{b}_j(x) = \left(\frac{\sin\frac{\tilde{\phi}}{2}(x+(j+2u_M)\mathrm{i})\sin\frac{\tilde{\phi}}{2}(x-j\mathrm{i})}{\sin\frac{\tilde{\phi}}{2}(x+j\mathrm{i})\sin\frac{\tilde{\phi}}{2}(x-(j-2u_M)\mathrm{i})}\right)^M. \tag{29}$$

Note that $\tilde{T}_0(x) = 1$ and $\boldsymbol{b}_j(x)$, $\tilde{T}_j(x)$ has poles at $x = 2nQ + u_M\mathrm{i} \pm (j - u_M)\mathrm{i}$ and $x = 2nQ + u_M\mathrm{i} \pm (j + 1 - u_M)\mathrm{i}$, respectively.

We define

$$Y_j(x) = \frac{\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x)}{\boldsymbol{b}_j(x)} \qquad j = 1, 2, \ldots. \tag{30}$$

For these functions the following relations stand:

$$\begin{aligned} &Y_1(x-\mathrm{i})Y_1(x+\mathrm{i}) = 1 + Y_2(x) \\ &Y_j(x+\mathrm{i})Y_j(x-\mathrm{i}) = (1+Y_{j-1}(x))(1+Y_{j+1}(x)) \qquad j = 2, 3, \ldots \\ &\lim_{l\to\infty}\frac{\ln Y_l(x)}{l} = \frac{2h}{T}. \end{aligned} \tag{31}$$

As $Y_j(x)$, $j = 2, 3, \ldots$, has no pole or zero in $-1 \leqslant \mathrm{Im}\, x \leqslant 1$, we find

$$\ln Y_j(x) = \boldsymbol{s} * (\ln(1 + Y_{j-1}) + \ln(1 + Y_{j+1})) \qquad j \geqslant 2. \tag{32}$$

For $Y_1(x)$ one must be careful that it has poles in $-\mathrm{i}$, $(1 - 2u_M)\mathrm{i}$. Using

$$\tilde{T}_2(x + \mathrm{i})\tilde{T}_2(x - \mathrm{i}) = \boldsymbol{b}_2(x)(1 + Y_2(x)) \tag{33}$$

and $\tilde{T}_2(x)$ has no zero or pole at $-1 \leqslant \mathrm{Im}\, x \leqslant 1$, we have

$$\ln \tilde{T}_2(x) = \boldsymbol{s} * (\ln \boldsymbol{b}_2(x) + \ln(1 + Y_2(x))). \tag{34}$$

Using $Y_1(x) = \tilde{T}_2(x)/\boldsymbol{b}_1(x)$ we have

$$\ln Y_1(x) = -\ln \boldsymbol{b}_1(x) + \boldsymbol{s} * \ln \boldsymbol{b}_2(x) + \boldsymbol{s} * \ln(1 + Y_2(x)). \tag{35}$$

In the limit of $M \to \infty$, the function $\boldsymbol{b}_j(x)$ can be simplified to

$$\begin{aligned}\boldsymbol{b}_j(x) &= \lim_{M\to\infty} \exp\left[M \ln \frac{\sin\frac{\tilde{\phi}}{2}(x + (j + 2u_M)\mathrm{i}) \sin\frac{\tilde{\phi}}{2}(x - j\mathrm{i})}{\sin\frac{\tilde{\phi}}{2}(x + j\mathrm{i}) \sin\frac{\tilde{\phi}}{2}(x - (j - 2u_M)\mathrm{i})}\right] \\ &= \exp\left(-\frac{2\pi J \sinh\phi}{\phi T}\boldsymbol{a}_j(x)\right) \qquad \boldsymbol{a}_j(x) \equiv \frac{\phi \sinh j\phi/(2\pi)}{\cosh j\phi - \cos(\phi x)}\end{aligned} \tag{36}$$

which has singularities at $x = 2nQ \pm j\mathrm{i}$. In the limit of $M \to \infty$ equations (35), (32), (31) are identical to (2). Substituting

$$\ln \tilde{T}_1(x) = \boldsymbol{s} * \ln[(1 + Y_1(x))/\boldsymbol{b}_1(x)] \tag{37}$$

into (19) we have (4). Then the Gaudin–Takahashi equations are derived from the QTM method (see also the treatment in [16, 17] for related models).

Consider the $M \to \infty$ limit of the functions $\boldsymbol{b}_j(x)$, $\tilde{T}_j(x)$ as

$$u_j(x) \equiv \lim_{M\to\infty} \tilde{T}_j(x). \tag{38}$$

Then from (28) we have the relation

$$u_1(x + \mathrm{i})u_1(x - \mathrm{i}) = \boldsymbol{b}_1(x) + u_2(x). \tag{39}$$

Note also the asymptotics $u_1(\pm\mathrm{i}\infty) = 2\cosh h/T$. We may assume the functions $u_1(x)$ and $u_2(x)$ have similar singularities at $x = 2nQ \pm 2\mathrm{i}$ and $x = 2nQ \pm 3\mathrm{i}$, respectively. If we write (39) as

$$u_1(x + \mathrm{i}) = \boldsymbol{b}_1(x)/u_1(x - \mathrm{i}) + u_2(x)/u_1(x - \mathrm{i}) \tag{40}$$

the LHS has singularities at $x = \mathrm{i}, -3\mathrm{i}$ in the fundamental region ($|\mathrm{Re}\, x| \leqslant Q$). The first term of the RHS has singularities at $x = \mathrm{i}, -\mathrm{i}, 3\mathrm{i}$ and the second term at $x = 3\mathrm{i}, -3\mathrm{i} - \mathrm{i}$. Then following the method in [13], we get an integral equation for $u_1(x)$,

$$\begin{aligned}u_1(x) = 2\cosh h/T + \oint_C \frac{\phi}{2}\left(\cot\frac{\phi}{2}[x - y - 2\mathrm{i}]\boldsymbol{b}_1(y + \mathrm{i}) + \cot\frac{\phi}{2}[x - y + 2\mathrm{i}]\boldsymbol{b}_1(y - \mathrm{i})\right) \\ \times \frac{1}{u_1(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}}.\end{aligned} \tag{41}$$

From the explicit expression of $u_1(x)$ (36), we see that the integral equation (41) is identical to the one obtained in [13]. The free energy is given by

$$f = -T \ln u_1(0). \tag{42}$$

4. Case $\Delta < 1$

In this case we have

$$\begin{aligned} &\mathsf{h}(u) = \sinh u \qquad \eta = \mathrm{i}\tilde{\theta}/2 \qquad \tilde{\theta} = \cos^{-1}\left(\frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)}\right) \\ &\alpha_M = \frac{\mathrm{i}}{2}\tanh^{-1}\left(\tan\tilde{\theta}\tanh\frac{J\Delta}{2MT}\right). \end{aligned} \tag{43}$$

In the limit of $M \to \infty$ we have

$$\tilde{\theta} = \cos^{-1}\Delta \qquad M\alpha_M = \mathrm{i}J\sin\theta/(4T). \tag{44}$$

Putting $x = \mathrm{i}v/\eta$ we obtain

$$Q(x) = \prod_{j=1}^{M}\sinh\frac{\tilde{\theta}}{2}(x - x_j) \qquad \varphi(x) = \left(\frac{\sinh\frac{\tilde{\theta}}{2}(x+\mathrm{i})\sinh\frac{\tilde{\theta}}{2}(x-(1-2u_M)\mathrm{i})}{\sin\tilde{\theta}\sin\tilde{\theta}(1-u_M)}\right)^M. \tag{45}$$

Kuniba *et al* [17] succeeded in deriving the Takahashi–Suzuki equations [18] for the thermodynamics of the *XXZ* model at $h = 0$, $|\Delta| < 1$. The functions

$$\tilde{T}_j(x) \equiv T_j(x)\left(\frac{\sin(\tilde{\theta})\sin\tilde{\theta}(1-u_M)}{\sinh\frac{\tilde{\theta}}{2}(x+(j+1)\mathrm{i})\sinh\frac{\tilde{\theta}}{2}(x-(j+1-2u_M)\mathrm{i})}\right)^M \tag{46}$$

are all periodic with periodicity $2p_0\mathrm{i}$. We have relations for $\tilde{T}_1(x)$ and $\tilde{T}_2(x)$

$$\tilde{T}_1(x+\mathrm{i})\tilde{T}_1(x-\mathrm{i}) = b_1(x) + \tilde{T}_2(x) \tag{47}$$

with

$$b_1(x) = \left(\frac{\sinh\frac{\tilde{\theta}}{2}(x+(1+2u_M)\mathrm{i})\sinh\frac{\tilde{\theta}}{2}(x-\mathrm{i})}{\sinh\frac{\tilde{\theta}}{2}(x+\mathrm{i})\sinh\frac{\tilde{\theta}}{2}(x-(1-2u_M)\mathrm{i})}\right)^M. \tag{48}$$

$\tilde{T}_1(x)$ satisfies

$$\tilde{T}_1(\pm\infty) = 2\cosh h/T. \tag{49}$$

By these two equations we can determine $\tilde{T}_1(x)$ in the limit of $M \to \infty$. In this limit $b_1(x)$ is

$$b_1(x) = \exp\left(-\frac{2\pi J\sin\theta}{\theta T}a_1(x)\right) \qquad a_1(x) \equiv \frac{\theta\sin\theta/(2\pi)}{\cosh(\theta x) - \cos\theta}. \tag{50}$$

We can assume that $\tilde{T}_1(x)$ is expanded as follows:

$$\tilde{T}_1(x) = 2\cosh\left(\frac{h}{T}\right) + \sum_{j=1}^{\infty}\sum_n \frac{c_j}{(x-2np_0\mathrm{i}-2\mathrm{i})^j} + \sum_{j=1}^{\infty}\sum_n \frac{\overline{c_j}}{(x-2np_0\mathrm{i}+2\mathrm{i})^j}. \tag{51}$$

Consider the contour integral around $x = \mathrm{i}$ giving the coefficients c_j

$$c_j = \oint \frac{(x-\mathrm{i})^{j-1}b_1(x)}{\tilde{T}_1(x-\mathrm{i})}\frac{\mathrm{d}x}{2\pi\mathrm{i}} = \oint \frac{y^{j-1}b_1(y+\mathrm{i})}{\tilde{T}_1(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}}. \tag{52}$$

The first sum of the RHS of (51) is

$$\begin{aligned} \sum_{j=1}^{\infty}\oint\sum_n \frac{b_1(y+\mathrm{i})}{(x-2np_0\mathrm{i}-2\mathrm{i})^j}\frac{y^{j-1}}{\tilde{T}_1(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}} &= \oint\sum_n \frac{b_1(y+\mathrm{i})}{x-y-2np_0\mathrm{i}-2\mathrm{i}}\frac{1}{\tilde{T}_1(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}} \\ &= \oint \frac{\theta}{2}\coth\frac{\theta}{2}(x-y-2\mathrm{i})\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y+\mathrm{i})\right]\frac{1}{\tilde{T}_1(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}}. \end{aligned} \tag{53}$$

The second sum is calculated in a similar way. Thus we find

$$u(x) = 2\cosh\left(\frac{h}{T}\right) + \oint_C \frac{\theta}{2}\left(\coth\frac{\theta}{2}[x-y-2\mathrm{i}]\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y+\mathrm{i})\right]\right.$$
$$\left. + \coth\frac{\theta}{2}[x-y+2\mathrm{i}]\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y-\mathrm{i})\right]\right)\frac{1}{u(y)}\frac{\mathrm{d}y}{2\pi\mathrm{i}} \qquad (54)$$

and the free energy is given by

$$f = -T\ln u(0). \qquad (55)$$

Apparently these equations are analytical continuations of (5) and (6) if we replace ϕ by $\mathrm{i}\theta$. Then equation (5) treats the thermodynamics in a unified way.

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